Percolation and self-avoiding random loops

LETTER TO THE EDITOR

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Abstract. We present simulations supporting the conjecture that properly defined exterior perimeters of 2D critical percolation clusters have the same fractal dimension, $D = \frac{4}{3}$, as self-avoiding random walks. In addition, we discuss possible generalizations.

Percolation and self-avoiding random walks are the two simplest and most important geometric models showing critical behaviour. Moreover, they are both related to thermal theories, both being equivalent to unphysical limits of spin models. Thus any new relationship between them would be very important. In this letter we conjecture that a very natural close connection between them exists indeed in two dimensions.

What we claim is essentially that perimeters of critical percolation clusters in two dimensions are random self-avoiding loops. To become non-trivial and correct, this statement needs several qualifying comments.

First of all, it is trivial that any perimeters are self-avoiding loops, and since percolation clusters are random objects, the above statement might seem altogether trivial. This is not the case since by 'random self-avoiding loops' we have a very specific measure on the space of all loops in mind. In this measure, all possible loops of a given fixed length N and not intersecting themselves have the same weight. This is the immediate generalization of the self-avoiding walk (SAW) measure, and the loops have the same scaling behaviour as the open walks: the average gyromagnetic radius of a N-step loop scales as N^ν, with $\nu = \frac{3}{4}$ in two dimensions [1]. Further scaling laws follow if we consider also loops with arbitrary N, and assume that every loop has still the same weight. Then, the number of loops scales as $c_N \sim \mu^N N^{-\gamma}$ with $\gamma = \frac{3}{2}$ [1], where the non-universal constant μ depends on the specific lattice.

Secondly, there are several possible definitions of the perimeter of a percolation cluster. The first problem is whether we count also the perimeters of internal holes, or only the exterior perimeter. We mean the latter. The best known definition of an exterior perimeter for percolation clusters is what is usually called its hull. This is defined as the set of all points which belong themselves to the cluster, but which are next to plaquettes belonging to the connected surrounding cluster on the dual lattice [2] (see also below). It is known that its dimension is exactly $\frac{7}{4}$ [3–8], larger than that of SAWs.

It was, however, observed by Grossman and Aharony [9] that a seemingly slight redefinition of the exterior perimeter, what they called the 'accessible' exterior perimeter, has a much smaller dimension close to $\frac{4}{3}$. More precisely they measured $D = 1.34 \pm 0.02$. Based on much more precise simulations we conjecture that D indeed is exactly $\frac{4}{3}$. The same conjecture had already been made in [7, 9]. Notice however that this conjecture is incompatible with unpublished measurements by Meakin and

Family (quoted in [7, 9]) which are supposed to show $D = 1.343 \pm 0.002$. Since we have no detailed information on these measurements, and since our own measurements described below have lower statistical errors, we will disregard these measurements in the following.

In [9], the accessible perimeter was only defined for site percolation in two dimensions, and was motivated by chemical considerations. Indeed, several definitions were given which all took into account the inaccessibility of narrow fjords in the cluster to objects as large or larger than the lattice constant a. It was found that the accessible perimeter dimension for site percolation on the square lattice decreased abruptly from $\frac{7}{4}$ to $\frac{4}{3}$ when these objects had diameter $> a$.

We found it more useful to use definitions which use the concept of the dual lattice. The dual lattice is obtained by exchanging sites with plaquettes. In this way, bond percolation on the square lattice and site percolation on the triangular lattice become self dual, while triangular lattice bond percolation becomes dual to honeycomb lattice bond percolation [10]. For these lattices, the thresholds are known exactly. On other lattices (like, e.g., site percolation on the square lattice studied mainly in [9]) the uncertainty of p_c introduces systematic errors in any simulation.

If we consider a finite cluster, then it is surrounded by one connected cluster on the dual lattice. This cluster will in general have some bond-dangling groups extending into the interior. These are groups of sites connected with the bulk only via paths passing through a single bond, see figure 1. This observation suggests immediately the first definition. The accessible perimeter $P^{(a)}$ of a cluster $\mathscr{C}$ is defined as those sites on the dual lattice which are in contact with $\mathscr{C}$, after removing all bond-dangling groups.
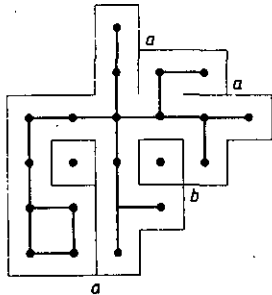

Figure 1. A finite bond percolation cluster on the square lattice (bold lines). The adjacent part of the surrounding cluster on the dual lattice is indicated by fine lines. It has a site-dangling group at b and bond-dangling groups at a.

A very similar definition is obtained by removing in addition all site-dangling groups, i.e. groups connected with the bulk only via paths passing through a single site (see figure 1). We call this the accessible perimeter $P^{(b)}$.

Notice that cutting off dangling groups means essentially that we consider only the backbone of the dual cluster. Thus, the accessible perimeter dimension of the original cluster is just the dimension of the backbone perimeter of the dual cluster. Simulations of the latter are given in [11], with the result $D = 1.344 \pm 0.022$. This is compatible with our more precise value quoted below.

Alternative definitions of accessible perimeters are suggested by the observation [3, 4, 6, 8, 12, 13] that the hulls of percolation clusters can be obtained by special random walks with memory. These are easiest for self-dual cases [4, 13]. For bond

percolation on the square lattice, this walk is particularly simple (see figure 2). It is a walk from bond to neighbouring bond, with right or left turn at every step, and with memory: if the walk comes again to a bond it has already visited before, then it has to take a turn in the same direction. On bonds not visited before (including the bond on which the walk had started), the chance for left and right turns is equal. The walk is continued until a loop is formed, which will finally happen with probability one.

It is easily seen that the statistics of such walks is just that of critical percolation hulls, except for a trivial factor taking into account the bias implied by the fact that the randomly chosen starting point has to belong to the hull. More precisely, the hull as defined above is the set of lattice sites on the interior side of any link touched by the walk.

Consider now a hull walk as constructed above, and remove from it all interior parts by always taking the turn which follows its outermost path (see figure 2(b); similar constructions work on other 2D lattices). This gives a walk which follows precisely the accessible perimeter $P^{(a)}$. We call it $\tilde{P}^{(a)}$. In order to get a walk (called $\tilde{P}^{(b)}$) which follows $P^{(b)}$, we would have in addition to remove all fjords whose entrance is formed by two parallel steps on the same plaquette of the original lattice (figure 2(c)).

In order to measure their perimeters, we first constructed hulls of critical bond percolation clusters by walks on a square lattice of bonds as described above. In order to avoid any finite lattice bias, we worked on a lattice of 2500×2500 bonds, and discarded any walk moving a distance >1250 away from the centre. For the remaining walks we constructed the perimeters $P^{(a)}$ and $P^{(b)}$. Their diameters, defined as

$$R = [(x_{max} - x_{min})^2 + (y_{max} - y_{min})^2]^{1/2} \tag{1}$$
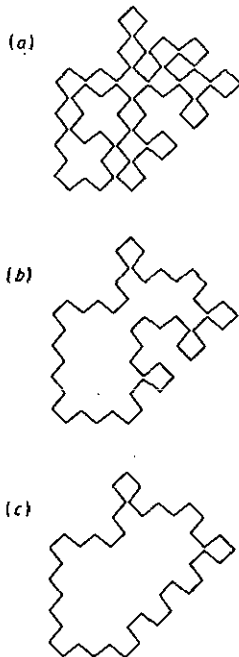

Figure 2. (a) Walk generating the hull of the percolation cluster of figure 1. (b) The perimeter $\tilde{P}^{(b)}$ obtained by always taking outer moves. (c) The perimeter $\tilde{P}^{(a)}$ obtained by deleting in addition fjords whose entrance is formed by two neighbouring parallel steps.

were used to bin the obtained perimeter lengths N. More precisely, in order to reduce finite-size effects, the ratios $N/R^{4/3}$ were binned. For perimeter $P^{(a)}$ the total number of clusters obtained in this way was 1.8×10^6, representing a sample more than two orders of magnitude larger than that of [10].

Let us call $r_i = \langle N/R^{4/3} \rangle_i$ the average over the ith bin. In figure 3 we show the local exponents

$$D_i = \frac{4}{3} + \frac{\log r_{i+1}/r_{i-1}}{\log R_{i+1}/R_{i-1}} \tag{2}$$

against $1/R_i$, where R_i is the centre of the ith bin. We see quite strong corrections to scaling, but they seem to have an exponent bigger than 1. Thus we can make a straight extrapolation to $R = 0$. Using only data with $R > 100$ for the extrapolation, we obtain $D = 1.332 \pm 0.002$.
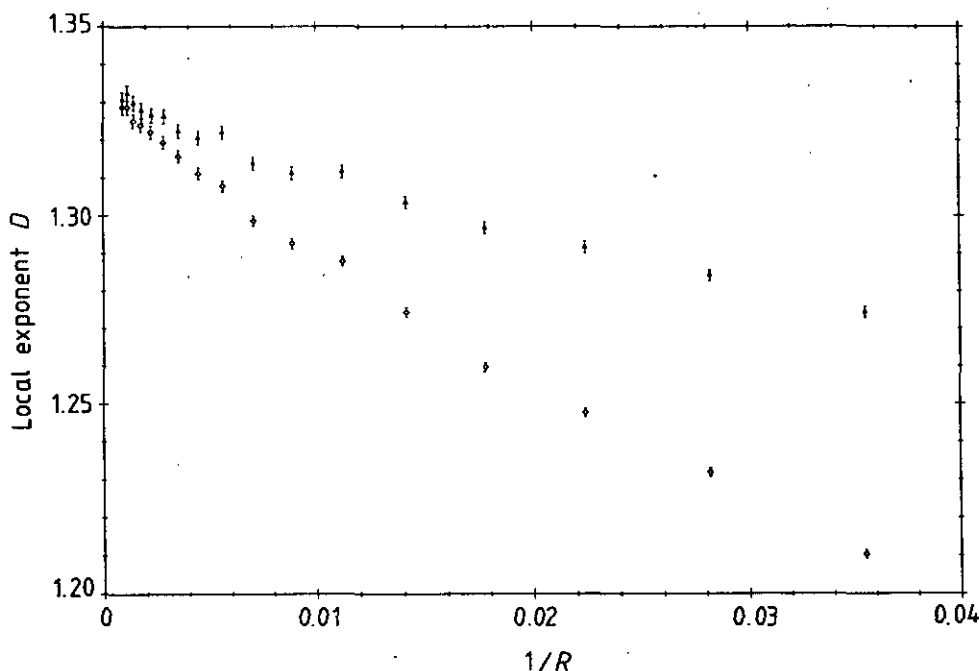

Figure 3. Local exponents D_i against the inverse average radius R_i. The extrapolation $R_i \to \infty$ gives the fractal dimension of the accessible perimeter. Triangles: $\tilde{P}^{(a)}$; diamonds: $\tilde{P}^{(b)}$.

We have not been able to give a good *a priori* reason why percolation cluster perimeters should be *random self-avoiding loops*. Notice that our truncations of dangling groups of dual clusters is not symmetric with respect to duality, as we do not truncate in a similar way the original cluster. Thus, we expect any relationship between SAWs and percolation perimeters to be true only in the scaling limit.

A possible deeper relationship between percolation and self-avoiding loops which encompasses not only the fractal dimension could be the following. Consider a large domain $\mathcal{D}_0$, say for simplicity a square of size $L \times L$. Place on it a loop $\mathcal{L}_1$ chosen randomly with the only constraint that it should fit entirely into $\mathcal{D}_0$. Since large loops

are more frequent, this loop will be typically of size $\sim L$. The complement $\mathscr{D}_0 - \mathscr{L}_1$ consists of two disjoint components, namely the exterior and the interior of the loop. We call it $\mathscr{D}_1$, and we place into one of its components another random loop which we call $\mathscr{L}_2$, etc. In a recursive way we generally place a random loop $\mathscr{L}_k$ in one of the k disjoint components of the domain $\mathscr{D}_{k-1}$ and call the resulting complement $\mathscr{D}_k$. We conjecture that the loops constructed in this way are in the scaling limit perimeters of critical percolation clusters.

Finally, let us discuss whether this can be extended to higher dimensions. An obvious conjecture would be that surfaces of critical percolation clusters are self-avoiding closed surfaces. This would be a very interesting result in view of the recent interest in self-avoiding surfaces coming mainly from non-Abelian gauge theories. An obvious problem with such a conjecture would be that these surfaces are topologically very complicated. They are in general not singly connected, and it is not clear whether this is the ensemble of main interest. Another problem is that at present there seems to be no efficient algorithms available by which such a conjecture could be tested numerically.

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